Retirement Planning Using Random Walk Theory and Artificial Intelligence

Tushar Sen
Chief Manager - Investments,
Metlife Insurance Co Ltd.
India.
E-mail: tushar.sen@gmail.com

Abstract: An investor more often than not seeks a right direction rather than obnoxious returns on his portfolio, and the biggest challenge for an investment advisory/fund management firm is to provide a clear direction in this chaotic market. An investor seeks to build his wealth over a period of time through rational investing and avoiding rash speculation, and for this he would need a healthy advice from time to time. An intelligent system built with robust inputs from a team of financial experts would be an ideal tool to help an investor make better judgments and avoid losing capital to speculation or sentimental portfolio adjustments. This paper provides a base upon which such an intelligent system can be built to provide sound investment advice to millions of clients who invest with one important purpose in mind – retirement planning. The paper attempts at building a financial retirement planning model using advanced mathematical techniques. The paper starts by developing the formal mathematics of the lognormal random walk model, Central Limit Theorem is applied to the model to argue that under three strong assumptions the values of risky investments at any time horizon are lognormally distributed. After a thorough understanding of a client's current financial status, financial needs and financial goals the inputs are fed into an enhanced random walk model developed for retirement planning. The enhanced random walk model is designed to accommodate monthly savings, varying time period, varying mean and standard deviation (arising out of varying asset allocation depending on client profile), time to retirement and initial portfolio value. As the enhanced random walk model gets tougher to provide a deterministic algorithm after accommodating several input variables we use Monte Carlo simulation technique to perform simulations to the tune of 105 and above to in order to obtain the

distribution of unknown probabilistic entity that then is read from a cumulative density function graph that plots probabilities against the desired retirement goal. The model includes periodic additions to or withdrawals from a portfolio, salary growth, inflation, investment expenses, and asset allocation among cash, bonds and stocks using artificial intelligence technique (fuzzy logic). Financial service institutions face a difficult task in evaluating clients risk tolerance. It is a major component for the design of an investment policy and understanding the implication of possible investment options in terms of safety and suitability. Instead of trying to build conventional mathematical models to arrive at a suitable asset allocation, task almost impossible when complex phenomena are under study, the presented methodology creates fuzzy logic models reflecting a given situation in reality and provide solution leading to suggestion for action. A rigorous architecture is worked out using fuzzy logic controls to arrive at a risk tolerance ability of a client based on his annual income and total networth, this new parameter is fed into another fuzzy logic control model that takes into account the age of the investor as well and exposures into stocks and bonds as a function of his risk tolerance ability and his age and is designed to prepare an appropriate asset allocation for the investor into the three basic asset classes: stocks, bonds and cash. The yearly returns of each of the three asset classes are meticulously calculated to provide utmost accuracy to the results of the model, the calculations are provided as appendix for the curious readers. The cumulative density function graph provides an estimate of the success of the retirement planning chalked out in the paper, it is assumed that a client would want the chances of his meeting his retirement goal should be atleast 50%, a figure lower than this would mean inadequate savings and then the parameters fed into the enhanced random walk model need to be looked at again and reworked and re-simulated. Another important aspect we looked at is the standard of living of a client and his willingness to lower down his standard of living post his retirement, his flexibility determines the kind of relative standard of living he wishes to adopt post retirement and that would determine his withdrawal rate which in turn would determine the survival rate of his portfolio. We discuss the limitations of this model and future scope of work on the base provided by the paper, the assumptions made in the preparation of the model and their impact in the results thus obtained.

Keywords: *Random Walk Theory, Lognormal Distribution, Retirement Portfolio Planning, Asset Allocation, Artificial Intelligence, Fuzzy Logic, Monte Carlo Method, Stochastic Differential Equations*

---

1 Introduction

We develop a lognormal random walk model for retirement planning. We start by discussing continuous compounding for risk-free investments.

Suppose we have s_0 dollars and invest it in a savings account or other risk-free investment earning a yearly interest rate of μ. What is the value s1 of our investment at the end of one year?

$$s_1 = s_0+s_0\mu = s_0(1+\mu) \qquad \text{(Eq1.1)}$$

In general, if interest is compounded n times per year, we have:

$$s_1 = s_0(1+\mu/n)^n \qquad \text{(Eq1.2)}$$

What happens if interest is compounded more and more frequently? In other words, what happens as n gets larger and larger in equations (1) and (2)? In the limit, we have:

$$s_1 = s_0\lim_{n\to\infty}\left(1+\frac{\mu}{n}\right)^n \qquad \text{(Eq1.3)}$$

$$r= \lim_{n\to\infty}\left(1+\frac{\mu}{n}\right)^n - 1 \qquad \text{(Eq1.4)}$$

To evaluate the limits in equations (3) and (4) we use L'Hôpital's rule. Let $x = 1/n$. Then:

$$\lim_{n\to\infty}\left(1+\frac{\mu}{n}\right)^n = e^{\lim_{x\to\infty} log\ (1+\mu x)^{\frac{1}{x}}}$$

$$\lim_{x\to\infty} log\ (1+\mu x)^{1/x} = \lim_{n\to\infty}(\mu/(1+\mu x)^n = \mu$$

$$\lim_{n\to\infty}\left(1+\frac{\mu}{n}\right)^n = e^{\mu}$$

Thus our equations (Eq1.3) and (Eq1.4) become:

$$s_1 = s_0 e^{\mu} \qquad \text{(Eq1.5)}$$

$$r = e^{\mu} - 1 \qquad \text{(Eq1.6)}$$

$$\mu = \log(r+1) \qquad \text{(Eq1.7)}$$

These equations tell us what happens if interest is compounded continuously, at every instant of time over the year. This is called *continuous compounding* and μ is called the *continuously compounded rate of return* of our investment. Suppose we have a risk-free investment s with an initial value of s_0 that earns a continuously compounded rate of return μ.

Let: s(t) = the value of the investment at time t

Then:

$$s(t) = s_0 e^{\mu t}$$

Consider the value of the investment a short time later, at time $t + dt$:

$$s(t+dt) = s_0 e^{\mu(t+dt)} = s_0 e^{\mu t} e^{\mu t} = s(t)\, e^{\mu t}$$

Let:

ds(t) = the growth of the investment over the time interval [t, $t + dt$]

Then:

$$ds(t) = s(t + dt) - s(t)$$

$$= s(t)\, e^{\mu dt} - s(t)$$

$$= s(t)\, (e^{\mu dt} - 1)$$

$$ds(t)/dt = e^{\mu dt} - 1$$

This equation holds at all times t, so we have the following differential equation which describes the behavior of our risk-free investment with continuous compounding:

$$ds/s = e^{\mu dt} - 1 \qquad \text{Eq(1.8)}$$

1.2 Risky Assets

In the previous section we examined risk-free investments that earn interest continuously over time. In this section we turn our attention to risky investments where the change in value of the investment over time is uncertain.

Let S be a risky investment with initial value S_0.

Consider a small time interval *dt* and let S_1 be the value of our initial investment S_0 after *dt* time has passed. Over this short time interval the rate of return of our investment is some random variable Y1, and the value S_1 of our investment at the end of the time interval is:

$$S_1 = S_0(1 + Y_1)$$

Now consider a second small time interval *dt*. Let S_2 be the value of our investment at the end of the second time interval, and let Y_2 be the random variable for the rate of return of our investment over the second time interval.

Then:

$$S_1 = S_0(1 + Y_1) = S_0(1 + Y_1)(1 + Y_2)$$

As time goes on, over each small time interval *dt* the value of our investment changes by some small random amount. Let S_n be the value of our investment at the end of *n* time intervals, and let Y_i be the random variable for the rate of return of our investment over the time interval *i*.

Then:

$$S_n = S_0\prod_{i=1}^{n}(1 + Yi)$$

Take the logarithm of both sides of this equation :

$$\log(S_n) = \log\left(S_0\prod_{i=1}^{n}(1 + Yi)\right)$$

$$= \log(S_0) + \sum_{i=1}^{n}(1 + Yi)$$

$$\log(S_n/S_0) = \log(S_n) - \log(S_0)$$

$$= \sum_{i=1}^{n}(1 + Yi)$$

For each *i*, let Z_i be the random variable $\log(1+Y_i)$. Then our equation becomes:

$$\log(S_n/S_0) = \sum_{i=1}^{n}(Zi) \qquad \text{(Eq1.9)}$$

1.3 Lognormal Random Walks

The equation (Eq1.9) which we derived in the previous section is not very useful without additional information about the distribution of the random variables Y_i which give the rate of return of the investment over time interval *i*.

We now make three strong assumptions about these random variables:

1. The random variables Y*i* are *independent*. What happens at one time interval does not affect what happens at subsequent time intervals. The market has no memory."
2. The random variables Y*i* are *identically distributed*. The means, standard deviations, and other attributes of the probability distributions do not change over time.
3. The random variables Y*i* have finite variance.

Recall equation (Eq1.9) from the previous section:

$$\log(S_n/S_0) = \sum_{i=1}^{n}(Zi) \text{ where } Z_i = \log(1 + Y_i)$$

If the random variables Y_i are independent, identically distributed, and have finite variance, then so do the random variables Z_i.

The Central Limit Theorem of Probability Theory says that in the limit, as $n \rightarrow \infty$, the average of *n* independent identically distributed random variables with finite variance is normally distributed. Thus under our three assumptions, we have:

$$\lim_{n\rightarrow\infty}\left(\frac{1}{n}\right) \log(S_n/S_0) \text{ is normally distributed}$$

Our time interval *dt* is very short. For example, if *dt* is one second, and each Y_i is the random rate of return of our investment over one second, then over a single seven hour trading day, we have n = 7 × 60 × 60 = 25, 200 seconds. It is reasonable to assume at this point that $\log(S_n/S_0)$ is normally distributed after even just one day. Indeed, it is reasonable to assume that $\log(S_n/S_0)$ is normally distributed for all *n*, with each Z_i normally distributed with identical means and variances.

Let:

$$\mu = E(Z_i)/dt$$

$$\sigma = Var(Z_i)/dt$$

Then Z_i is $N[\mu dt, \sigma^2 dt]$. Define:

$$dX_i = (Z_i - \mu dt)/\sigma$$

Then:

$$Z_i = \mu dt + \sigma dX_i \qquad \text{where } dX_i \text{ is } N[0,dt]$$

Define:

$$s(t) = \text{the value of the investment at time } t$$

and let:

$$n = t/dt$$

Then:

$$\log(s(t)/s(0)) = \log(s_n/s_0)$$

$$= \sum_{i=1}^{n}(Zi)$$

$$= \sum_{i=1}^{n}(\mu dt + \sigma dXi) \text{ where } dX_i \text{ is } N[0,dt]$$

$$= \sum_{i=1}^{n} \mu dt + \sum_{i=1}^{n} \sigma dXi$$

$$= n\mu dt + \sigma \sum_{i=1}^{n} dXi$$

$$= \mu t + \sigma \sum_{i=1}^{n} dXi$$

The variables dX_i are independent normally distributed random variables and each has mean 0 and variance dt.$\sum_{i=1}^{n} dXi$also is normally distributed and has mean 0 and variance $n \times dt = t$.

Thus we have:

$$\log(s(t)/s(0)) = \mu t + \sigma X_i \qquad \text{where } X \text{ is } N[0,dt] \qquad \text{(Eq1.10)}$$

$$s(t)/s(0) = e^{\mu t + \sigma X i} \qquad \text{(Eq1.11)}$$

$$s(t) = s(0)e^{\mu t + \sigma X i} \qquad \text{(Eq1.12)}$$

Note that $\mu t + \sigma X$ is normally distributed $N[\mu dt, \sigma^2 dt]$, so s(t)=s(0) is lognormally distributed $LN[\mu dt, \sigma^2 dt]$.

When t = 1 (one year) we have:

$$s(1) = s(0)e^{\mu + \sigma X} \text{ where } X \text{ is } N[0,1]$$

$$s(1) = s(0)e^{\dot{X}} \text{ where } \dot{X} \text{ is } N[0,1] \qquad \text{(Eq1.13)}$$

Finally, consider the change in value *ds(t)* of the investment s over a short time interval [*t*, *t* + *dt*]. We have:

$$s(t+dt) = s(t)e^{\mu dt + \sigma dX} \qquad \text{where } dX \text{ is } N[0,dt]$$

$$ds(t) = s(t+dt) - s(t)$$

$$ds(t)/s(t) = s(t)e^{\mu dt + \sigma dX} - 1$$

This equation holds at all times *t*, so we have the following stochastic differential equation which describes the behavior of our risky investment over time:

$$ds/s = e^{\mu dt + \sigma dX} - 1 \text{ where } \qquad dX \text{ is } N[0,dt] \qquad \text{(Eq1.14)}$$

Compare this equation (Eq1.14) to the ordinary differential equation (Eq1.8) we derived in for the behavior of a risk-free investment over time with continuous compounding:

$$ds/s = e^{\mu dt} - 1 \qquad \text{(Eq1.15)}$$

The difference between these equations is that equation (Eq1.14) has the additional random term $\sigma dX$to account for the uncertainty (riskiness) of our investment. Equation (Eq1.15) is the special case of equation (Eq1.14) when $\sigma = 0$.

Equation (Eq1.14) is one formulation of *the lognormal random walk model*. μ is the *continuously compounded expected rate of return* of the investment, and σ is the *standard deviation of the continuously compounded returns*.

2 Client’s details for retirement planning

Suppose Elvis is 25 years old. He currently earns a gross salary of $40,000 per year. He has a defined contribution retirement program at work where he contributes 10% of his gross salary via monthly payroll deduction and his employer adds a generous 1-for-1 matching contribution of 10% of his gross salary. Elvis can contribute more than the standard 10% if he wishes (he doesn't currently), but he doesn't get any matching contribution from his employer for any such additional savings. Elvis's retirement portfolio has been growing for a few years and has a current market value of $250,000.

Elvis tells us that he would like to retire at age 60 and have enough money so that his standard of living in retirement will be the same as his standard of living in his last year of work prior to retirement. He wants his income during retirement to keep up with inflation. His only source of income will be his retirement portfolio savings from his job. Elvis wants to know what his chances are of achieving his goal. How might we go about trying to help Elvis figure out his problem using a random walk model we developed?

3 Developing the Model

We have four main problems to solve in developing a random walk model to help Elvis plan for his retirement:

1. How much money does Elvis need to accumulate in order to retire with his stated goals?
2. How do we calculate his asset allocation?
3. How do we estimate the parameters μ and σ for the random walk model?
4. How do we enhance the model to accommodate Elvis's monthly savings and his employer's matching contribution?
5. How do we compute the density and cumulative density functions for the enhanced random walk model?

Let's solve these problems in order.

3.1 Problem 1: How Much Money Do We Need?

We look at Elvis's paycheck stubs and discover that he is paying the typical 35% of his gross salary in Loans and 5% towards life/health insurance premiums. In addition, Elvis is currently putting away 10% of his gross salary as retirement savings, which is another expense he will not have when he retires. So Elvis only needs to replace 50% of his last year's gross salary to maintain his net yearly income in retirement. Note that we do not subtract federal or state income taxes in this analysis, since Elvis will still have to pay those taxes when he retires. For simplicity, we assume that Elvis will be in the same tax bracket. This seems to be a reasonable assumption since his income will be about the same. We have now determined that Elvis's retirement portfolio must be large enough to replace 50% of his gross salary when he retires in order to maintain his standard of living.

Let x = 50% of Elvis's gross salary in his last year of work prior to retirement. This is the amount Elvis withdraws from his portfolio during his first year of retirement. In his second year he adjusts x for inflation and withdraws another x dollars. In his third year he adjusts x again, withdraws another x dollars, and so on throughout his retirement. How big does Elvis's retirement portfolio have to be to support these yearly inflation-adjusted withdrawals of x dollars? This is a complicated topic which we will discuss in detail later. For now we make the assumption that 3% is a reasonable inflation-adjusted withdrawal rate. Thus Elvis needs to accumulate 1/3% = 33 times x dollars in his retirement portfolio. x is 50% of Elvis's last year's gross salary. So Elvis needs to accumulate33 times 60% = 16.67~17 times his last year's gross salary. This solves our first problem.

3.2 Problem 2: How do we calculate his asset allocation?

We use fuzzy logic controls to work out two things:

a) Elvis’s risk tolerance ability based on his annual income and total networth

b) Elvis’s asset allocation based on his risk tolerance ability and age

First we present a simple model of client's *risk tolerance ability* which depends on his/hers *annual income (AI)* and *total networth (TNW)*.The control objective of the client financial risk tolerance policy model is for any given pair of input variables (*annual income, total networth*) to find a corresponding output, a *risk tolerance (RT)* level. Suppose the financial experts agree to describe the input variables *annual income* and *total networth* and the output variable *risk tolerance* by the sets

Annual Income ≜**A** = $\{A_1, A_2, A_3\}$ = {**L**, **M**, **H**}

Total Networth≜**B** = $\{B_1, B_2, B_3\}$ = {**L**, **M**, **H**}

Risk Tolerance ≜**C** = $\{C_1, C_2, C_3\}$ = {**L**, **MO**, **H**}

Hence the number of terms in each term set is $n = m = l = 3$. The terms have the following meaning: **L**≜*low*, **M**≜medium, **H***high*, **MO**≜*moderate*. They are fuzzy numbers whose supporting intervals belong to the universal sets $U_1 = \{\varkappa \text{ x } 10^3 | 0 \leq \varkappa \leq 100\}$, $U_2 = \{y \text{ x } 10^3 | 0 \leq y \leq 100\}$, $U_3 = \{z \text{ x } 10^3 | 0 \leq z \leq 100\}$. The real numbers $\varkappa$ and y represent dollars in thousands and hundreds of thousands, correspondingly, while z takes values on a psychometric scale from 0 to 100 measuring risk tolerance. The numbers on that scale have specified meaning for the financial experts.

The terms of the linguistic variables *annual income*, *total networth*, and *risk tolerance* described by triangular and part of trapezoidal numbers formally have the same membership functions presented analytically below:

Eq (3.1)

$$\mu L(v) = \begin{cases} 1 \quad for \ \ 0 \leq v \leq 20 \\ \dfrac{50 - v}{30} for \ \ 20 \leq v \leq 50 \end{cases}$$

$$\mu M(v) = \begin{cases} \dfrac{v - 20}{30} for \ \ 20 \leq v \leq 50 \\ \dfrac{80 - v}{30} for \ \ 50 \leq v \leq 80 \end{cases}$$

$$\mu H(v) = \begin{cases} \frac{v-50}{30} for\ \ 50 \leq v \leq 80 \\ 1 \qquad for\ \ 80 \leq v \leq 100 \end{cases}$$

Next step is setting the *if ... and ... then* rules of inference called also *control rules* or *production rules*. The number of the rules is *nm*, the product of the number of terms in each input linguistic variable ***A*** and ***B***. For the client financial risk tolerance model *n = m = l = 3*. Hence the number of *if ... then* rules is 9 and the number of different outputs is 3. Assume that the financial experts selected the rules presented on the decision below.

	Total Networth ***B***			
Annual Income*A*		**L**	**M**	**H**
	L	**L**	**L**	**MO**
	M	**L**	**MO**	**H**
	H	**MO**	**H**	**H**

The rules have as a conclusion the terms in the output C (see Eq3.1).

They read:

Rule 1: *If client's annual income (CAI) is low (***L***) and client's total networth (CTN) is low (***L***), then client's risk tolerance (CRT) is low(***L***);*

Rule 2: *If CAI is* ***L*** *and CTN is medium (***M***), then CRT is* ***L****;*

Rule 3: *If CAI is* ***L*** *and CTN is high (***H***), then CRT is moderate (***MO***);*

Rule 4: *If CAI is* ***M*** *and CTN is* ***L****, then CRT is* ***L****;*

Rule 5: *If CAI is* ***M*** *and CTN is* ***M****, then CRT is* ***MO****;*

Rule 6: *If CAI is* ***M*** *and CTN is* ***H****, then CRT is* ***H****;*

Rule 7: *If CAI is* ***H*** *and CTN is* ***L****, then CRT is* ***MO****;*

Rule 8: *If CAI is* ***H*** *and CTN is* ***M****, then CRT is* ***H****;*

Rule 9: *If CAI is* ***H*** *and CTN is* ***H****, then CRT is* ***H****.*

Here following Mamdani (1975) we define the rule of inference as a conjunction-based rule expressed by operation $\wedge$ (min); r_k is called conclusion or consequent. Hence we have

$p_i \wedge q_j \wedge r_k = \min(\mu Ai(x), \mu Bj(y), \mu Cij(z))$, $r_k = r_{ij}$;

i = 1,….,n; j = 1,….,m; k = 1,….,l; and $(x; y; z) \in A \times B \times C \subseteq U_1 \times U_2 \times U_3$.

Rule 1: $p_1 \wedge q_1 \wedge r_{11} = \min(\mu L(x), \mu L(y), \mu L(z))$,

Rule 2: p1 ∧ q2 ∧ r12 = $\min(\mu L(x), \mu M(y), \mu L(z))$,

Rule 3: p1 ∧ q3 ∧ r13 = $\min(\mu L(x), \mu H(y), \mu MO(z))$,

Rule 4: p2 ∧ q1 ∧ r21 = $\min(\mu M(x), \mu L(y), \mu L(z))$,

Rule 5: p2 ∧ q2 ∧ r23 = $\min(\mu M(x), \mu M(y), \mu MO(z))$,

Rule 6: p2 ∧ q3 ∧ r23 = $\min(\mu M(x), \mu H(y), \mu H(z))$,

Rule 7: p3 ∧ q1 ∧ r31 = $\min(\mu H(x), \mu L(y), \mu MO(z))$,

Rule 8: p3 ∧ q2 ∧ r32 = $\min(\mu H(x), \mu M(y), \mu H(z))$,

Rule 9: p3 ∧ q3 ∧ r33 = $\min(\mu H(x), \mu M(y), \mu H(z))$,

Elvis's readings: $\varkappa_0$ = 40in thousands (*annual income*) and y_0 = 25 in ten of thousands (*total networth*). The fuzzy inputs are calculated from (Eq3.1). Note that $\varkappa$ = 40 and y = 25 are substituted for v instead of 40,000 and 250,000 since $\varkappa$ and y are measured in thousands and ten of thousands. The result is

$\mu L(40) = 1/3$, $\mu M(40) = 2/3$, $\mu L(25) = 5/6$, $\mu M(25) = 1/6$

For $\varkappa = \varkappa_0 = 40$ and $y = y_0 = 25$ the decision Table reduces to an induced Table

Table 3.1 - Induced decision table for the Elvis's financial risk tolerance model.

	$\mu L(25) = 5/6$	$\mu L(25) = 5/6$	0
$\mu L(40) = 1/3$	$\mu L(z)$	$\mu L(z)$	0
$\mu M(40) = 2/3$	$\mu L(z)$	$\mu MO(z)$	0
0	0	0	0

There are four active rules, 1,2,4,5 and the strength of these rules (the and part) is calculated as follows:

$\alpha_{11} = \mu L(40) \wedge \mu L(25)$ = min (1/3, 5/6) = 1/3,

$\alpha_{12} = \mu L(40) \wedge \mu M(25)$ = min (1/3,1/6) = 1/6,

α_{21}= $\mu M(40) \wedge \mu L(25)$ = min (2/3,5/6) = 2/3

α_{22} = $\mu M(40) \wedge \mu M(25)$ = min (2/3,1/6) = 1/6

These results are presented in the rules strength Table.

Table 3.2 - Rules strength table for the client's financial risk tolerance model.

	$\mu L(25)$ = 5/6	$\mu L(25)$ = 5/6	0
$\mu L(40)$ = 1/3	1/3	1/6	0
$\mu M(40)$ = 2/3	2/3	1/6	0
0	0	0	0

For the control outputs (CO) of the rules we have

CO of rule 1 :$\alpha_{11} \wedge \mu L(z)$ = min (1/3, $\mu L(z)$),

CO of rule 2 : $\alpha_{12} \wedge \mu L(z)$ = min (1/6, $\mu L(z)$),

CO of rule 3 :$\alpha_{21} \wedge \mu L(z)$ = min (2/3, $\mu L(z)$),

CO of rule 4 :$\alpha_{22} \wedge \mu MO(z)$ = min (1/6, $\mu MO(z)$),

The result concerning only the active cells is given on Table 5.8.

Table 3.3 – Control Output table

. . .	. . .	. . .	. . .
. . .	$1/3 \wedge \mu L(z)$	$1/6 \wedge \mu L(z)$	. . .
. . .	$2/3 \wedge \mu L(z)$	$1/6 \wedge \mu MO(z)$	. . .
. . .	. . .	. . .	. . .

The output of the four control rules now have to be aggregated in order to produce one control output with membership function $\mu agg(z)$. It is natural to use for aggregation the operator ∨ (*or*) expressed by max:

$\mu agg(z)$ = max ($1/3 \wedge \mu L(z)$, $1/6 \wedge \mu L(z)$, $2/3 \wedge \mu L(z)$, $1/6 \wedge \mu MO(z)$)

Now, for a real number α and a fuzzy set C with a membership function $\mu C(z)$, we define

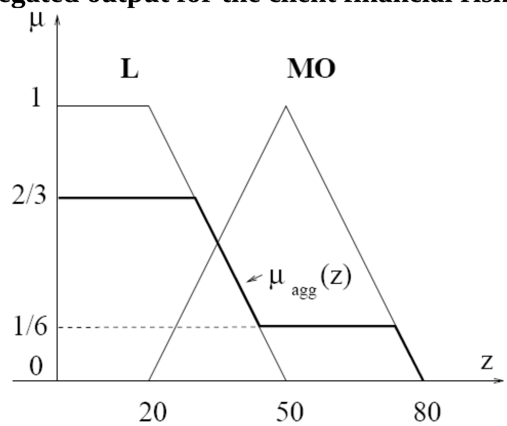
$\alpha \wedge \mu C(z)$ = min (α_{11}, $\mu C(z)$)

Therefore the aggregated output of the control rules is:

$\mu agg(z)$ = max {min (2/3, $\mu L(z)$), min(1/6,$\mu MO(z)$)}

is geometrically represented in Figure 3.1

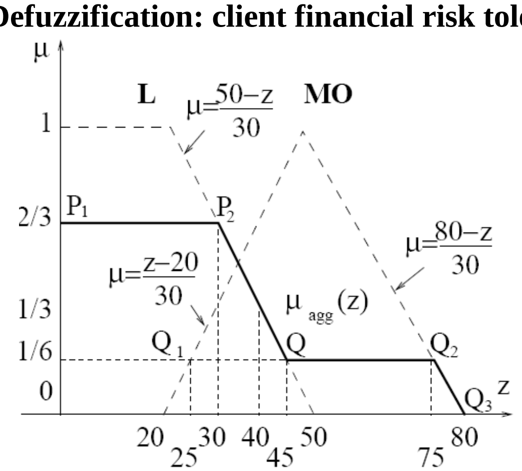
Figure 3.1- Aggregated output for the client financial risk tolerance model.



Let us defuzzify the aggregated output for the client financial risk tolerance model using Height defuzzification method (HDM). First we express analytically the aggregated control output with membership function $\mu agg(z)$ shown on Figure 3.12. It consists of the four segments P1P2, P2Q, QQ2, and Q2Q3 located on the straight lines μ = 2/3 and μ = (50 – z)/30, μ = 1/6 and μ = (90 – z)/10, correspondingly. Solving together the appropriate equations gives the projections of P_2, Q, Q_2 on z axis, namely 30, 45, 75 (Figure 3.2). They are used to specify the domains of the segments forming $\mu agg(z)$. Hence

$$\mu agg(z) = \begin{cases} \frac{2}{3} \; for \; 0 \le z \le 30 \\ \frac{50-z}{30} \; for \; 30 \le z \le 45 \\ \frac{1}{6} \; for \; 45 \le z \le 75 \\ \frac{80-z}{30} \; for \; 75 \le z \le 80 \end{cases}$$

Figure 3.2 - Defuzzification: client financial risk tolerance model.



HDM is a generalization of mean of maximum method. Besides the segment P_1P_2 with height p there is another at segment Q_1Q_2 with lower height q. The midpoint of the interval $[\eta_1, \eta_2]$, the projection of Q_1Q_2 on z, is $(\eta_1+\eta_2)/2$. Then the HDM produces $\hat{z}_h$:

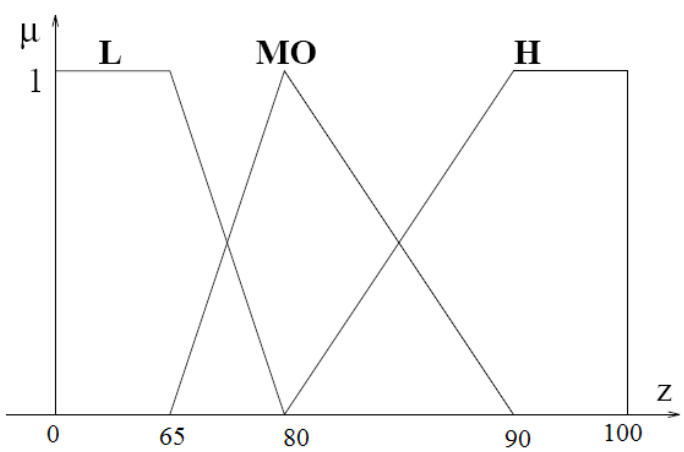
$$\hat{z}_h = \{ p(\zeta_1 + p\zeta_2)/2 + q(\eta_1+\eta_2)/2 \}/ (p + q) = w_1(\zeta_1 + \zeta_2)/2 + w_2 (\eta_1+\eta_2)/2 \quad \text{(Eq3.2)}$$

Substituting $\mu = 1/6$ into $\mu = (z - 20)/30$ gives the number 25, the projection of the point Q_1. Hence the flat segments P_1P_2 and Q_1Q_2 in Fig have projections [0,30] and [25,75] and heights 2/3 and 1/6 correspondingly, i.e. $\zeta_1 = 0$, $\zeta_2 = 30$, $\eta_1 = 25$, $\eta_2 = 75$, $p = 2/3$, $q = 1/6$. The result of substituting these values in HDM is

$\hat{z}_h = 22$.

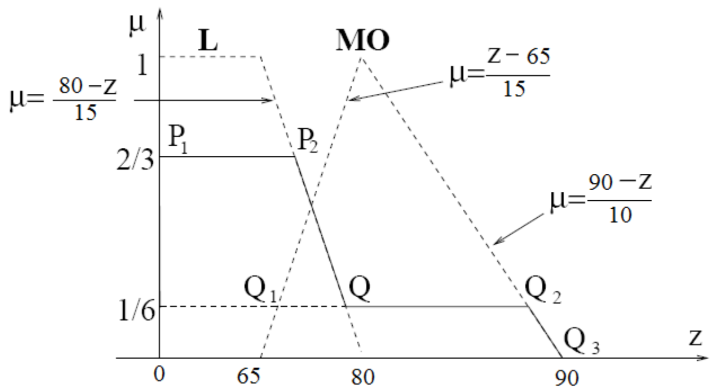
The financial experts could estimate the clients financial risk tolerance given that his annual income is $40,000 and total networth is $250,000 to be 22 on a scale from 0 to 100 if they adopt the HDM. Accordingly a conservative risk investment strategy could be suggested. However, the conclusion of the FLC model, namely the crisp value 22(HDM) measuring the risk tolerance on the scale from 0 to 100 to be too small for a person with annual income 40,000 and total networth 250,000. Hence we need to fine-tune the model making slight change to the terms of output C-*risk tolerance*. The modified terms are shown on Figure 3.3.

Figure 3.3 – Aggregated Output



In comparison to Figure 3.1 there are several changes: (1) The new terms **L** and **H** have new supporting intervals [0, 65] instead of [0, 50] and [65, 100] instead of [50, 100], correspondingly; (2) the new term **MO** has its peak shifted to the left by 15 units; it is still a triangular number but not in central form. Assuming everything else in the model in stays without change, firing of the same rules produces here the aggregated output given in Figure 3.4.

Figure 3.4 – Aggregated Output



Solving together $\mu = 2/3$ and $\mu = (80 - z)/15$, $\mu = 1/6$ and $\mu = (z - 65)/15$, $\mu = 1/6$ and $\mu = (90 - z)/10$ we find that the projections of P_1P_2 and Q_1Q_2 are [0,70] and [67.5, 88.33] respectively.

Therefore using the HDM method the non fuzzy control output is

$\hat{z}_h \approx 45$.

Therefore our new risk tolerance has increased to 45.

We now go on to define the asset allocation of Elvis based on his risk tolerance hence arrived at, and for this we need to model another FLC where the inputs (linguistic variables) in the fuzzy logic client asset allocation model are *age* and *risk tolerance* (risk) and there are three outputs(linguistic variables), *savings, income, and equity*. Nevertheless the technique in the earlier model can be applied but that requires the design of three decision tables. The control objective is for any given pair (age, risk) which reflects the state of a client to find how to allocate the asset to *savings, income, and growth*.

Assume that the financial experts describe the two input and three output variables by the terms of triangular and trapezoidal shape as follows:

Age≙ {**Y** *(young)*, **MI** *(middle age)*, **OL** *(old)*}

Risk≙ {**L** *(low)*, **MO** *(moderate)*, **H***(high)*}

Saving≙ {**L** *(low)*, **M***(medium)*, **H** *(high)*}

Income≙ {**L** *(low)*, **M***(medium)*, **H** *(high)*}

Growth≙ {**L** *(low)*, **M***(medium)*, **H** *(high)*}

The universal sets (operating domains) of the input and output variables are$U_1 = \{\varkappa \mid 0 \le \varkappa \le 100\}$, $U_2 = \{y \mid 0 \le y \le 100\}$, $U_3 = \{z_i \mid 0 \le z_i \le 100, i = 1,2,3\}$. The real numbers $\varkappa$ represents years and y takes values on a psychometric scale from 0 to 100 and z_i take values on scale from 0 to 100.The terms of linguistic variables risk, savings, income, and growth are described by the same membership functions as the linguistic variables in the previous model Eq 2.a. The variable age differs slightly from the other variables; the membership functions of its terms are:

(Eq3.3)

$$\mu\boldsymbol{Y}(x) = \begin{cases} 1 & for \;\; x \le 20 \\ \dfrac{45 - v}{25} & for \;\; 20 \le x \le 45 \end{cases}$$

$$\mu\boldsymbol{MI}(x) = \begin{cases} \dfrac{x - 20}{25} & for \;\; 20 \le x \le 45 \\ \dfrac{70 - x}{25} & for \;\; 45 \le x \le 70 \end{cases}$$

$$\mu\boldsymbol{OL}(x) = \begin{cases} \dfrac{x - 45}{25} & for \;\; 45 \le x \le 70 \\ 1 & for \;\; 70 \le x \end{cases}$$

There are nine *if ... and ... then* rules like in the previous model but each inference rule produces three (not one) conclusions, one for savings, one for income, and one for growth. Consequently the financial experts have to design three decision tables. Assume that these are the tables presented below.

Table3.4 - Decision table for the output *savings*.

	Risk Tolerance			
Age		*Low*	*Moderate*	*High*
	Young	**M**	**L**	**L**
	Middle	**M**	**L**	**L**
	Old	**H**	**M**	**M**

Table 3.5 - Decision table for the output *income*.

	Risk Tolerance			
Age		*Low*	*Moderate*	*High*
	Young	**M**	**M**	**L**
	Middle	**H**	**H**	**M**
	Old	**H**	**H**	**M**

Table 3.6 - Decision table for the output *growth*.

	Risk Tolerance			
Age		*Low*	*Moderate*	*High*
	Young	**M**	**M**	**L**
	Middle	**H**	**H**	**M**
	Old	**H**	**H**	**M**

For instance the first two *if ... then* rules read:

If client's age is young and client's risk tolerance is low, then asset allocation is: medium in savings, medium in income, medium in growth.

If client's age is young and client's risk tolerance is moderate, then asset allocation is: low in savings, medium in income, high in growth.

Consider Elvis whose age is $\varkappa_0 = 25$ and risk tolerance level as calculated earlier is$y_0 = 45$. Matching the readings 25 and 45 using Eqs. (5.3) and (6.1) gives the fuzzy reading inputs

$\mu\boldsymbol{Y}(25) = 4/5$, $\mu\boldsymbol{MI}(25) = 1/5$, $\mu\boldsymbol{L}(45) = 1/6$, $\mu\boldsymbol{MO}(45) = 5/6$

The strength of the rules calculated using (5.10) are:

$\alpha_{11} = \mu\boldsymbol{Y}(25) \wedge \mu\boldsymbol{L}(45)$ = min (4/5, 1/6) = 1/6,

$\alpha_{12} = \mu\boldsymbol{Y}(25) \wedge \mu\boldsymbol{MO}(45)$ = min (4/5,5/6) = 4/5,

α_{21}= $\mu\boldsymbol{MI}(25)\wedge\mu\boldsymbol{L}(45)$ = min (1/5,1/6) = 1/6,

α_{22} = $\mu\boldsymbol{MI}(25)\wedge\mu\boldsymbol{MO}(45)$ = min (1/5,5/6) = 1/5

The control outputs of the rules are presented in the active cells in three decision tables

Table 3.7- Control output savings

	Low	Moderate
Young	$1/6 \wedge\mu\boldsymbol{M}(z1)$	$4/5 \wedge\mu\boldsymbol{L}(z1)$
Middle	$1/6 \wedge\mu\boldsymbol{M}(z1)$	$1/5 \wedge\mu\boldsymbol{L}(z1)$

Table 3.8 - Control output income.

	Low	Moderate
Young	$1/6 \wedge\mu\boldsymbol{M}(z2)$	$4/5 \wedge\mu\boldsymbol{M}(z2)$
Middle	$1/6 \wedge\mu\boldsymbol{H}(z2)$	$1/5 \wedge\mu\boldsymbol{H}(z2)$

Table 3.9 - Control output growth.

	Low	Moderate
Young	$1/6 \wedge\mu\boldsymbol{M}(z3)$	$4/5 \wedge\mu\boldsymbol{H}(z3)$
Middle	$1/6 \wedge\mu\boldsymbol{L}(z3)$	$1/5 \wedge\mu\boldsymbol{M}(z3)$

The outputs in the four active cells in Tables 3.7-3.9 have to be aggregated separately. The results obtained by following the earlier model are:

$\mu agg(z1)$ = max {min (1/6, $\mu\boldsymbol{M}(z1)$), min(4/5,$\mu\boldsymbol{L}(z1)$)}

$\mu agg(z2)$ = max {min (4/5, $\mu\boldsymbol{L}(z2)$), min(1/5,$\mu\boldsymbol{H}(z2)$)}

$\mu agg(z3)$ = max {min (1/5, $\mu\boldsymbol{M}(z3)$), min(4/5,$\mu\boldsymbol{H}(z3)$, min(1/6,$\mu\boldsymbol{L}(z3)$)}

Figure 3.5 - Aggregated output savings. Defuzzification.

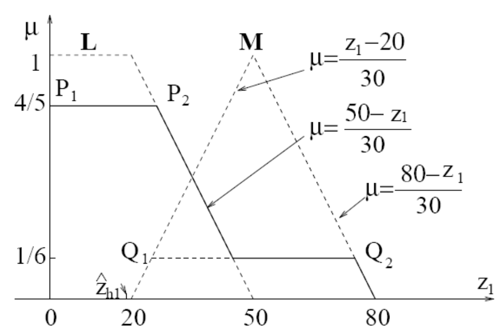


Figure 3.6 - Aggregated output savings. Defuzzification.

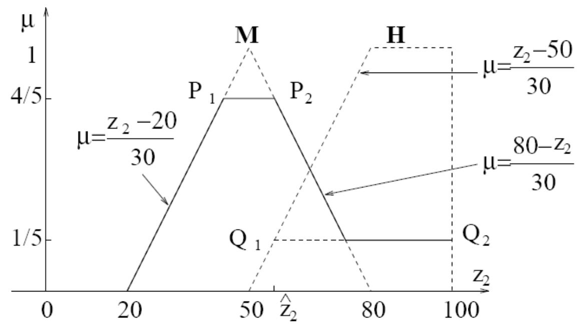
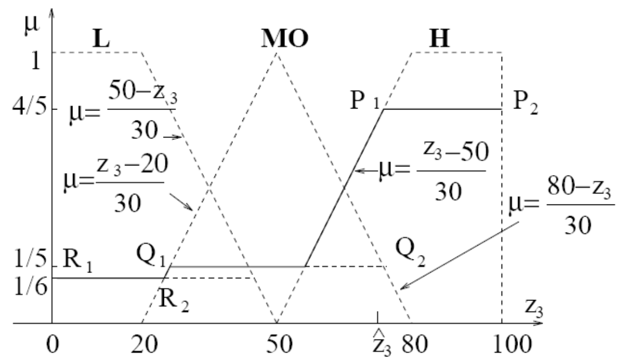


Figure 3.7 - Aggregated output savings. Defuzzification.



The aggregated outputs shown on Figures3.5 -3.7 are defuzzified by using HDM. The results are given in the same figures.

The projections of the flat segments can be easily found using their height and the relevant equations of inclined segments indicated in the figures. Substituting gives the projection of P_2 to be 26. Substituting gives the projections of Q_1 and Q_2 to be 25 and 75. Similarly one can find that the projections of P_1P_2 and Q_1Q_2 in Figure3.3 are the intervals [44,56] and [56, 100]. There are three at segments P_1P_2, Q_1Q_2, and R_1R_2 in Figure 3.4. Their projections are [74,100], [26, 74], and [0,45].

Then using the defuzzification formula (Eq3.2) we find

$\hat{z}_{h1} \approx 19$ (*saving*),

$\hat{z}_{h2} \approx 56$ (*income*),

$\hat{z}_{h3} \approx 71$ (*growth*),

The sum $\hat{z}_{h1} + \hat{z}_{h2} + \hat{z}_{h3}$ = 146.42 represents the total asset (100%). Rearranging this gives the following asset allocation of the client whose age is 25and risk tolerance 45:

Savings : 19.38% → 13.18%;

Income : 55.60% → 37.81%;

Growth : 71.44% → 48.58%;

Rounding off gives savings 13%, income 38%, and growth 49%. Therefore Elvis's asset allocation is decided upon based on his *risk tolerance*(45) and *age*(25).

3.3Problem 3: How Do We Estimate the Parameters?

How do we estimate the parameters μ and σ for the random walk model?

First we deal with salary raises and inflation. It may not be obvious what these issues have to do with parameter estimation, but that should become clear shortly.

A good way to model inflation and salary growth is to use real rates of return relative to Elvis's salary growth rather than nominal returns. So when we use our time series data to estimate the model parameters we need to apply the conversion formula given above. The next issue that affects parameter estimation is investment expenses. Brokers, mutual fund investment companies, insurance companies, and retirement plan management companies all charge fees. We do a bit of research and discover that Elvis's investment expenses in his retirement plan are 35 basis points (0.35%).To properly model Elvis's portfolio, when we do our parameter estimation, before converting from nominal to real returns, we must subtract 0.35% for Elvis's investment expenses.

To summarize, Elvis's yearly salary increase is 1.5% in excess of inflation. His investment expenses are35 basis points.His asset allocation is: Cash (*savings*) 13%, Bonds (*income*) 38%, and Stocks (*growth*) 49%. We're now ready to do the parameter estimation for our random walk model. We have the following four sets of historical market time series data from 2003 through 2013:

C = Cash (*saving*) = 90 day Treasury bills yearly returns

B = Bonds (*income*) = 10 year Govt of India bonds yearly returns

C = Stocks (*growth*) = CNX NIFTY index yearly returns

I = Inflation = Consumer Price Index percent change, the average inflation for the past 10 years is 6.64% and that is taken as a constant for calculations.

We construct a new time series N for the nominal return of Elvis's portfolio after expenses as follows:

N = 0.13× C + 0.38× B + 0.49× S – 0.0035 (Eq3.3)

We then convert from nominal returns to real returns relative to Elvis's salary growth:

R = (N – (I + 0.015)) / (1 + (I + 0.015)) (Eq3.4)

We finally estimate the parameters μ and σ for our model by converting to *continuous compounding* and finding the *mean* and *standard deviation*:

μ= E(log(1 + R)) = 0.06984

$\sigma = \sqrt{\mathrm{Var}(\log(1 + R))}$ = 0.15344

Note that the expected continuously compounded real rate of return after expenses for Elvis's portfolio relative to his salary growth is 6.98%, with rather large standard deviation of 15.33%.

3.4 Problem 4: How Do We Enhance the Model?

The random walk model we developed in the earlier section is:

$$ds/s = e^{\mu dt + \sigma dX} - 1 \text{ where } dX \text{ is } N[0,dt]$$

In this stochastic differential equation, *dt* is an infinitesimal time interval, *ds* is the change in the portfolio value over the time interval, and *dX* is a random variable with variance *dt*.

This equation has a useful property for computation. *dt* does not have to be an infinitesimal time interval. It can be any time interval. For example, the equation works fine with *dt* = one day, *dt* = one month, or *dt* = one year. We change notation to recognize this:

$$\Delta s/s = e^{\mu\Delta t + \sigma\Delta X} - 1 \text{ where } \Delta X \text{ is } N[0,\Delta t]$$

Unfortunately, this model does not deal with periodic savings or withdrawals from the portfolio. We have to add an extra term to accommodate this:

$$\Delta s/s = e^{\mu\Delta t + \sigma\Delta X} - 1 + (k\Delta t/s) \quad \text{where } \Delta X \text{ is } N[0,\Delta t] \qquad \text{(Eq3.5)}$$

$$s(t+\Delta t) = s(t)e^{\mu\Delta t + \sigma\Delta X} - 1 + k\Delta t \quad \text{where } \Delta X \text{ is } N[0,\Delta t] \qquad \text{(Eq3.6)}$$

Where k is the constant yearly extra amount added to ($k > 0$) or withdrawn from ($k < 0$) the portfolio in even installments at the end of each time period Δt.

Equations (Eq3.5) and (Eq3.6) are our enhanced random walk model for retirement planning. In Elvis's case he contributes 10% of his gross salary via monthly payroll deduction and his employer contributes a 10% matching contribution. Thus for Elvis's model we use $k = 0.20$ and $\Delta t = 1/12 = 0.08333$.

We now have a complete random walk model for Elvis's retirement plan. To summarize everything we have done so far, the model is given by equations (Eq3.5) and (Eq3.6) above with the following parameters:

$\mu = 0.06984$

$\sigma = 0.15344$

$s_0 = 5.0$

$k = 0.20$

$\Delta t = 1/12 = 0.8333$

$t = 35$

Elvis's goal is to accumulate a total of 17 times his salary, s(t) = 17. Elvis would like to see the cumulative density function for s(t). This would give him an answer to his primary question what are my chances of reaching my goal? This leads us to our last problem number 5.

3.5 Problem 5: How Do We Do the Computations?

How do we compute the density and cumulative density functions for the enhanced random walk model?

Let $n = t/\Delta t$ and $\Delta X i$ = independent $N\,[0, \Delta t]$ random variables for $i = 1 \dots n$.

It is a bit laborious but quite easy to show by induction that:

$$s(t) = s(0)\prod_{i=1}^{n} e^{\mu\Delta t + \sigma\Delta Xi} + \sum_{i=1}^{n-1} k\Delta t + \prod_{j=i+1}^{n} e^{\mu\Delta t + \sigma\Delta Xi} + k\Delta t \quad \text{(Eq3.7)}$$

The first term is the growth of our initial portfolio value over the full *t* year period.

The second term is the sum of the growth of each monthly savings amount $k\Delta t$ over the months from the time the savings are added to the portfolio through the end of the full period. The last term is the last month's savings, made at the end of the month.

We can rewrite this equation as:

$$s(t) = s(0)e^{\mu\Delta t + \sigma(\sum_{n=1}^{n}(\Delta Xi))} + k\Delta t\sum_{i=1}^{n-1} e^{\mu(n-i)\Delta t + \sigma\sum_{j=i+1}^{n}\Delta Xi} + k\Delta t$$

Can we simplify this equation further? Note that s(t) is a sum of *n* lognormally distributed random variables plus a constant, but the random variables are not independent, and even if they were, their sum would not be lognormally distributed. (The product of two independent lognormally distributed random variables is lognormally distributed, but their sum is not). We seem to be at an impasse. s(t) no longer has a simple distribution which we can easily compute, so our retirement savings graphs are going to be harder to calculate. How might we go about computing the density and cumulative density functions for s(t)?

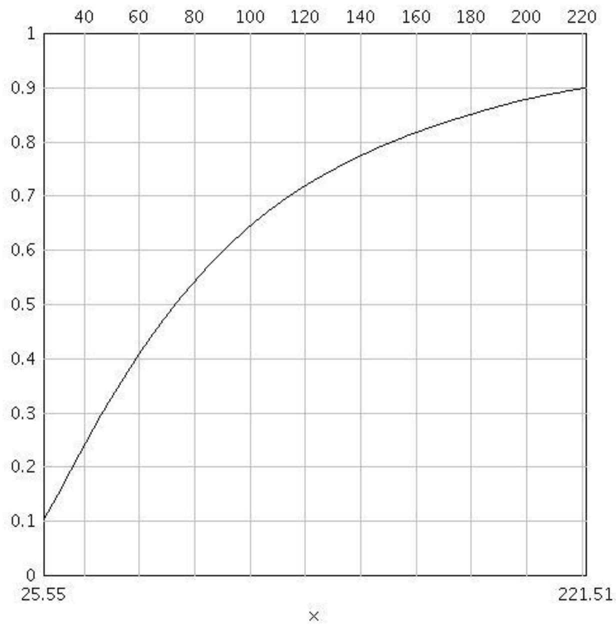
Our goal is to graph the density function and the cumulative density function for s(t). Suppose we simulate some large number of random walks in our enhanced model, say 100,000 of them, and record all the ending values. We end up with a long list of 100,000 ending values. To graph the density function, we partition the range of ending values into a number of evenly spaced buckets and count up the number of ending values in each bucket. We draw a histogram of the result. If our buckets are sufficiently narrow, say only one screen pixel wide in our graph on the screen, and if our number of samples is sufficiently large, the result is a close approximation to the density function graph. The cumulative density function is also easy to compute.

To estimate Probe($s(t) < k$) for any *k*, simply walk through the list and count up the number of elements which are less than k. The function value is the count divided by 100,000.Monte Carlo simulation is the technique we implement for our enhanced model. Fortunately, modern personal computers are so fast that this technique is quite feasible and works well in practice.

4 Model in action

Figure 4.1 shows the density and cumulative density functions for s(15) for Elvis's current retirement savings. From the cumulative density function, we see that Elvis will meet definitely meet his goal of s(15) = 17.

Figure 4.1 – 35 Year Cumulative Density Function



Elvis now has a 50/50 chance of meeting his goal. But what if the markets do worse or better than the median over the next 35 years? How do these possibilities affect Elvis's retirement plans? To help measure the impact of various possible outcomes, we introduce the notion of Elvis's relative standard of living in retirement. We define this to be the ratio of Elvis's net income after retirement to his net income in his last working year prior to retirement. Elvis's goal is to have this ratio be at least 1.0. A ratio less than 1.0 is bad. A ratio greater than 1.0 is good. Before he retires, Elvis's net income is 95%–x measured as a percentage of his gross salary (*he continues to pay 5% annually for his health insurance*). After he retires, his net income is his social security income plus the 3% withdrawal from his retirement portfolio. Let p be the value of Elvis's portfolio when he retires. Then his net income after retirement is $0.03p$, again measured as a fraction of his gross salary.

We divide to get:

Relative standard of living = 0.03*p* / (0.95–*x*)

Consider the 10th and 90th percentiles in the cumulative density graph in Figure 4.1. The ending values *p* are 25.55 and 221.51 respectively. We have *x* = 10% =0.1 (*his regular savings as discussed in the section 3*). Plugging in these values in our equation gives relative standard of living ratios of 0.90 and 7.80. Thus, if the markets do unusually poorly over the next 35 years (at the 10th percentile or worse), Elvis must accept 90% or less of his desired standard of living after retirement, assuming he is not willing to continue working after age 60. If Elvis is concerned about this possibility, which he probably should be, he may want to save a little bit more.

On the other hand, if the markets do well over the next 35 years (at the 90th percentile or better), Elvis can retire with 7.8 times his target standard of living, or even more. Elvis is quite thrilled at this prospect! In our analysis of Elvis's retirement plan we see once again that uncertainty rules the world of investing. Elvis is 35 years away from retirement, yet we can only give him an 80% chance that his post-retirement standard of living will be somewhere between 90% and 780% of his pre-retirement standard of living. This is a wide range of possible outcomes, and it only covers 80% of the possibilities!

There are several things Elvis could do to try to deal with this problem. First, Elvis could adopt a different retirement strategy; instead of withdrawing a constant inflation-adjusted amount every year from his portfolio, he could perhaps try to withdraw less than usual during years in which the market isn't doing well. A second thing Elvis could do is use all or part of his retirement savings when he retires to purchase a lifetime annuity from an insurance company. This is infact a popular decision made by many retirees. The insurance company guarantees a steady income stream for as long as Elvis lives.

5 Conclusion and assumptions and Possible Directions for Future Work

The Random Walk Model designed was able to deliver a retirement plan (derived from a cumulative density function graphed by running Monte Carlo simulations) based on current age of an investor, his current risk tolerance ability, his monthly savings towards retirement, his asset allocation (derived from a fuzzy logic control system) his current portfolio value, his current expenditures and his expected retirement goal. With such mathematical rigor we could provide the client with a relative standard of living ranging from .9 to 7.8 times. The model also allows an investor to choose his withdrawal rates to suit his relative standard of living.

We have made several strong simplifying assumptions about Elvis's retirement plan. We assumed that Elvis does not vary his asset allocation over the 35 years remaining until his planned retirement. This assumption is called constant relative risk aversion. This is a strong assumption. There is no reason to assume that all investors have constant relative risk aversion.

Our random walk model can be modified to accommodate these more complicated kinds of relative risk aversion. Instead of being constants, our parameters μ and σ become functions of the portfolio value s.

We also assumed that Elvis saves the same constant fraction of his gross salary every month. For some investors, as they get older and their children leave home and they pay off their mortgages (for example), they may find that they are able to save a much larger fraction of their gross salary for retirement. To accommodate these kinds of situations, in our model we must change the parameter k to be a function of time instead of a constant. We have also assumed that Elvis's future salary growth rate is some constant amount in excess of inflation. This assumption does not take into account the variability of this excess amount or allow for the modeling of salary growth rates which we expect to increase or decrease as Elvis gets older. It would be possible to enhance the model for these factors by making the salary growth rate in excess of inflation a random variable with its own estimated growth rate and volatility parameters. On an even more ambitious scale, it is certainly conceivable that we could enhance the model and the program to do a complete life-cycle plan. Such a model and simulation might, for example, combine the pre-retirement and post-retirement analysis, take into account Elvis's human capital during his working years as part of the risk aversion machinery, and accommodate his desire (if any) to leave a bequest for his heirs.

References

[1] Marek Capinski, Tomasz Zastawniak (2003).*Mathematics for Finance: An Introduction to Financial Engineering*. UK. Springer-Verlag

[2] Edwin J Elton, Martin Gruber (2006).*Modern Portfolio Theory and Investment Analysis*. NY. John Wiley & Sons

[3] Richard A Ferri (2008).*All About Asset Allocation*. NY. McGraw Hill

[4] Mamdani, E. H. and Assilian, S. (1975) *An Experiment in Linguistic Synthesis with a Fuzzy Logic Controller*, Int. Journal Man-Machine Studies 7, pp. 1-13.

[5] Terano, T., Asai, K., and Sugeno, M. (1992) *Fuzzy Systems Theory and its Applications*, Academic Press, Boston.

[6] Zvi Bodie, Alex Kane, and Alan J. Marcus. *Investments*. Irwin McGraw-Hill, fourth edition, 1999.

[7] Eugene F. Fama. Random walks in stock market prices. Financial Analysts Journal, Sep-Oct 1965.

[8] Burton G. Malkiel. *A Random Walk Down Wall Street*. W. W. Norton, sixth edition, 1996.

Appendix I – Yearly Return Calculations of Bonds & Stocks

The yields of 10 Year Govt of India Bonds are fetched from database, the difference between the yields at the beginning of the year and end of the year is then multiplied by the Duration to arrive at the yearly returns of the GOI Bonds.

Dates	Yield	Beginning	Ending	Difference in Yield	Duration	(A) x (B)	Return
12/31/1998	12.22%			(A)	(B)		
12/31/1999	11.22%	12.22%	11.22%	-0.99%	6.03	5.98%	18.20%
12/28/2000	10.95%	11.22%	10.95%	-0.28%	6.25	1.72%	12.95%
12/31/2001	7.94%	10.95%	7.94%	-3.01%	6.31	18.97%	29.92%
12/31/2002	6.07%	7.94%	6.07%	-1.87%	7.08	13.24%	21.18%
12/31/2003	5.13%	6.07%	5.13%	-0.94%	7.63	7.16%	13.23%
12/31/2004	6.55%	5.13%	6.55%	1.42%	7.94	-11.29%	-6.16%
12/30/2005	7.11%	6.55%	7.11%	0.56%	7.49	-4.19%	2.36%
12/29/2006	7.61%	7.11%	7.61%	0.50%	7.32	-3.64%	3.48%
12/31/2007	7.79%	7.61%	7.79%	0.18%	7.17	-1.30%	6.31%
12/31/2008	5.26%	7.79%	5.26%	-2.53%	7.12	18.02%	25.81%
12/31/2009	7.59%	5.26%	7.59%	2.33%	7.90	-18.38%	-13.12%
12/31/2010	7.92%	7.59%	7.92%	0.33%	7.18	-2.37%	5.22%
12/30/2011	8.57%	7.92%	8.57%	0.65%	7.08	-4.59%	3.33%
12/31/2012	8.05%	8.57%	8.05%	-0.52%	6.90	3.57%	12.14%
12/31/2013	8.83%	8.05%	8.83%	0.77%	7.05	-5.46%	2.59%

Appendix II – Portfolio Return Calculation

Salary Growth	1.5%	Bonds = 10 year Govt Bonds
Bond Allocation	38%	Stocks = CNX NIFTY
Stock Allocation	49%	Cash = 90 days Govt T-Bill
Cash Allocation	13%	CPI = consumer price index
Std deviation	15.34%	
Mean	6.98%	

	Yearly returns				Nominal Return				Real continuous
Year	**Cash**	**Bonds**	**Stocks**	**CPI (change)**	**Cash**	**Bonds**	**Stocks**	**Portfolio**	**Portfolio**
2003	4.23%	13.23%	71.90%	6.64%	-7.02%	-1.53%	17.74%	32.66%	28.26%
2004	4.78%	-6.16%	10.68%	6.64%	-6.95%	-10.32%	-3.77%	-1.83%	-1.84%
2005	5.33%	2.36%	36.34%	6.64%	-6.89%	-6.46%	5.24%	12.86%	12.10%
2006	6.32%	3.48%	39.83%	6.64%	-6.77%	-5.95%	6.47%	14.95%	13.94%
2007	7.09%	6.31%	54.77%	6.64%	-6.67%	-4.67%	11.72%	22.81%	20.54%
2008	7.67%	25.81%	-51.79%	6.64%	-6.61%	4.17%	-25.73%	-18.55%	-20.52%
2009	3.54%	-13.12%	75.76%	6.64%	-7.10%	-13.47%	19.10%	25.07%	22.37%
2010	5.01%	5.22%	17.95%	6.64%	-6.92%	-5.16%	-1.22%	5.49%	5.35%
2011	7.71%	3.33%	-24.62%	6.64%	-6.60%	-6.02%	-16.18%	-14.14%	-15.24%
2012	8.36%	12.14%	27.70%	6.64%	-6.52%	-2.03%	2.21%	12.74%	11.99%
2013	8.23%	2.59%	6.76%	6.64%	-6.54%	-6.36%	-5.15%	-0.12%	-0.12%

Appendix III - Yearly Calculation of 90 days Govt T-Bill

The annualized yields of 90 days T-Bill are divided by 4 to arrive at a quarterly yield and then the quarterly yields over 4 quarters are added to arrive at the yearly returns.

Dates	Annual	Quarterly	Cumulative
11/27/2003	4.23	4.23	4.23
2/23/2004	4.325	1.08125	4.77875
5/22/2004	4.38	1.095	
8/21/2004	4.825	1.20625	
11/18/2004	5.585	1.39625	
2/16/2005	5.225	1.30625	5.325
5/17/2005	5.165	1.29125	
8/16/2005	5.135	1.28375	
11/14/2005	5.775	1.44375	
2/10/2006	6.575	1.64375	6.3225
5/12/2006	5.625	1.40625	
8/10/2006	6.36	1.59	
11/8/2006	6.73	1.6825	
2/6/2007	7.3	1.825	7.09375
5/7/2007	7.525	1.88125	
8/6/2007	6.35	1.5875	
11/2/2007	7.2	1.8	
2/1/2008	7.05	1.7625	7.665
5/2/2008	7.285	1.82125	
7/30/2008	9.25	2.3125	
10/29/2008	7.075	1.76875	

Dates	Annual	Quarterly	Cumulative
1/27/2009	4.625	1.15625	3.5375
4/27/2009	3.125	0.78125	
7/24/2009	3.225	0.80625	
10/23/2009	3.175	0.79375	
1/21/2010	3.8	0.95	5.0125
4/21/2010	4.25	1.0625	
7/20/2010	5.45	1.3625	
10/18/2010	6.55	1.6375	
1/17/2011	7.1	1.775	7.71
4/15/2011	7.16	1.79	
7/15/2011	8.1	2.025	
10/13/2011	8.48	2.12	
1/11/2012	8.44	2.11	8.355
4/10/2012	8.75	2.1875	
7/9/2012	8.2	2.05	
10/8/2012	8.03	2.0075	
1/7/2013	8.1	2.025	8.229375
4/5/2013	7.8247	1.956175	
7/4/2013	7.4678	1.86695	
10/3/2013	9.525	2.38125	
12/31/2013	8.625	2.15625	

Years	Cumulative Annual Yield
2003	4.23%
2004	4.78%
2005	5.33%
2006	6.32%
2007	7.09%
2008	7.67%
2009	3.54%
2010	5.01%
2011	7.71%
2012	8.36%
2013	8.23%